

Supersymmetry and discrete transformations on S^1 with point singularities

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Abstract

We investigate N -extended supersymmetry in one-dimensional quantum mechanics on a circle with point singularities. For any integer n , $N = 2n$ supercharges are explicitly constructed and a class of point singularities compatible with supersymmetry is clarified. Key ingredients in our construction are n sets of discrete transformations, each of which forms an $su(2)$ algebra of spin $1/2$. The degeneracy of the spectrum and spontaneous supersymmetry breaking are briefly discussed.

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1 Introduction

Quantum mechanics in one dimension admits point singularities as interactions of zero range. A point singularity is parameterized by the group $U(2)$ [1, 2, 3], and the parameters characterize connection conditions between a wavefunction and its derivative at the singularity. The variety of the connection conditions leads to various interesting physical phenomena, such as duality [4, 5], the Berry phase [6, 7], scale anomaly [8] and supersymmetry [9, 10, 11, 12]. Since a system with a number of point singularities possesses a wider parameter space, the system can have new features. In this Letter, we show that 2^n point singularities on a circle can realize $N = 2n$ supersymmetry.

In Ref. [9], $N = 2$ supersymmetry was discussed in the system of a free particle on a line \mathbf{R} or an interval $[-l, l]$ with a point singularity. In Ref. [11], this work was extended to $N = 4$ supersymmetry in the system on a pair of lines \mathbf{R} or intervals $[-l, l]$ each having a point singularity. In Ref. [10], two point singularities were put on a circle, and $N = 2$ supersymmetric models with a superpotential were constructed. Spectral properties and domains of operators in a supersymmetry algebra on a circle with a singularity are discussed in Ref. [12].

The purpose of this Letter is to examine N -extended supersymmetry in quantum mechanics on a circle with point singularities. Since it is hard to extend the work of Ref. [11] to higher N -extended supersymmetry, we follow the approach of Ref. [10] to realize $N = 2n$ supersymmetry for any integer n . A key ingredient in the analysis of Ref. [10] is a set of the discrete transformations that forms an $su(2)$ algebra of spin 1/2. In the next section, we extend it to n sets of discrete transformations, each of which forms an $su(2)$ algebra of spin 1/2 and commutes with the others. Since these transformations, in general, make wavefunctions discontinuous at some points on a circle, point singularities are inevitable in our construction of N -extended supersymmetry.

In Section 2, we introduce n sets of discrete transformations, each of which forms an $su(2)$ algebra of spin 1/2. In Section 3, we construct $N = 2n$ supercharges in terms of these transformations. In Section 4, we find a class of connection conditions compatible with supersymmetry. In Section 5, the degeneracy of the spectrum and spontaneous supersymmetry breaking are briefly discussed. Section 6 is devoted to summary and discussions.

2 Discrete transformations

The system we consider is one-dimensional quantum mechanics on a circle $S^1(-l < x \leq l)$ with 2^n point singularities placed at

$$x = l_s \equiv \left(1 - \frac{s}{2^{n-1}}\right) l, \quad \text{for } s = 0, 1, \dots, 2^n - 1. \quad (2.1)$$

In this setup, we are allowed to introduce n sets of discrete transformations $\{\mathcal{P}_k, \mathcal{Q}_k, \mathcal{R}_k\}$ ($k = 1, 2, \dots, n$) which produce singularities at $x = l_s$ for $s = 0, 1, \dots, 2^n - 1$. The \mathcal{P}_k ($k = 1, 2, \dots, n$) are a kind of the parity transformation, and the action of \mathcal{P}_k on an arbitrary function $\varphi(x)$ is defined by

$$(\mathcal{P}_k\varphi)(x) = \sum_{s=1}^{2^{k-1}} \Theta\left(x - \left(1 - \frac{s}{2^{k-2}}\right)l\right) \Theta\left(\left(1 - \frac{s-1}{2^{k-2}}\right)l - x\right) \times \varphi\left(-x + \left(2 - \frac{2s-1}{2^{k-2}}\right)l\right), \quad \text{for } k = 1, 2, \dots, n, \quad (2.2)$$

where $\Theta(x)$ is the Heaviside step function. The \mathcal{P}_1 is just a familiar parity transformation, $(\mathcal{P}_1\varphi)(x) = \varphi(-x)$. An example of the action of \mathcal{P}_3 on a function $\varphi(x)$ is given in Fig. 1. We see that the action of \mathcal{P}_3 , in general, produces singularities at $x = 0, \pm \frac{l}{2}$ and l . The action of \mathcal{P}_k for $k = 1, 2$ and 3 is schematically depicted in Fig. 2. The \mathcal{R}_k ($k = 1, 2, \dots, n$) are a kind of the half-reflection transformation, and the action of \mathcal{R}_k on $\varphi(x)$ is defined by

$$(\mathcal{R}_k\varphi)(x) = \sum_{s=1}^{2^{k-1}} (-1)^s \left[-\Theta\left(x - \left(1 - \frac{s-1/2}{2^{k-2}}\right)l\right) \Theta\left(\left(1 - \frac{s-1}{2^{k-2}}\right)l - x\right) + \Theta\left(x - \left(1 - \frac{s}{2^{k-2}}\right)l\right) \Theta\left(\left(1 - \frac{s-1/2}{2^{k-2}}\right)l - x\right) \right] \varphi(x), \quad \text{for } k = 1, 2, \dots, n. \quad (2.3)$$

For $k = 1, 2$ and 3, $(\mathcal{R}_k\varphi)(x)$ are explicitly given by

$$(\mathcal{R}_1\varphi)(x) = \begin{cases} +\varphi(x) & \text{for } 0 < x < l, \\ -\varphi(x) & \text{for } -l < x < 0, \end{cases} \quad (2.4)$$

$$(\mathcal{R}_2\varphi)(x) = \begin{cases} +\varphi(x) & \text{for } -l < x < -\frac{l}{2} \text{ and } \frac{l}{2} < x < l, \\ -\varphi(x) & \text{for } -\frac{l}{2} < x < \frac{l}{2}, \end{cases} \quad (2.5)$$

$$(\mathcal{R}_3\varphi)(x) = \begin{cases} +\varphi(x) & \text{for } -l < x < -\frac{3l}{4}, -\frac{l}{4} < x < \frac{l}{4} \text{ and } \frac{3l}{4} < x < l, \\ -\varphi(x) & \text{for } -\frac{3l}{4} < x < -\frac{l}{4} \text{ and } \frac{l}{4} < x < \frac{3l}{4}, \end{cases} \quad (2.6)$$

and are schematically depicted in Fig. 3. The third transformations \mathcal{Q}_k ($k = 1, 2, \dots, n$) are defined, in terms of \mathcal{P}_k and \mathcal{R}_k , by

$$\mathcal{Q}_k \equiv -i\mathcal{R}_k\mathcal{P}_k \quad \text{for } k = 1, 2, \dots, n. \quad (2.7)$$

Important observations are that each set of $\{\mathcal{P}_k, \mathcal{Q}_k, \mathcal{R}_k\}$ ($k = 1, 2, \dots, n$) forms an $su(2)$ algebra of spin 1/2, i.e.,

$$\begin{aligned} \mathcal{P}_k\mathcal{Q}_k &= -\mathcal{Q}_k\mathcal{P}_k = i\mathcal{R}_k, \\ \mathcal{Q}_k\mathcal{R}_k &= -\mathcal{R}_k\mathcal{Q}_k = i\mathcal{P}_k, \\ \mathcal{R}_k\mathcal{P}_k &= -\mathcal{P}_k\mathcal{R}_k = i\mathcal{Q}_k, \\ (\mathcal{P}_k)^2 &= (\mathcal{Q}_k)^2 = (\mathcal{R}_k)^2 = 1, \end{aligned} \quad (2.8)$$

and

$$[\mathcal{O}_k, \mathcal{O}_{k'}] = 0 \quad \text{if } k \neq k' \quad (2.9)$$

for $\mathcal{O}_k \in \{\mathcal{P}_k, \mathcal{Q}_k, \mathcal{R}_k\}$ and $\mathcal{O}_{k'} \in \{\mathcal{P}_{k'}, \mathcal{Q}_{k'}, \mathcal{R}_{k'}\}$.

For later use, let us introduce new sets of $su(2)$ generators $\{\mathcal{G}_{\mathcal{P}_k}, \mathcal{G}_{\mathcal{Q}_k}, \mathcal{G}_{\mathcal{R}_k}\}$ ($k = 1, 2, \dots, n$) as

$$\mathcal{G}_{\vec{\mathcal{P}}_k} = \mathcal{V}_k^\dagger \vec{\mathcal{P}}_k \mathcal{V}_k, \quad \text{for } k = 1, 2, \dots, n \quad (2.10)$$

for $\mathcal{V}_k = e^{i\vec{v}_k \cdot \vec{\mathcal{P}}_k} \in SU(2)$. Here, we have used $\vec{\mathcal{P}}_k$ as an abbreviation of $\vec{\mathcal{P}}_k = (\mathcal{P}_k, \mathcal{Q}_k, \mathcal{R}_k)$. The new $su(2)$ generators $\mathcal{G}_{\vec{\mathcal{P}}_k}$ have to be linearly related to $\vec{\mathcal{P}}_k$ as

$$\begin{aligned} \mathcal{G}_{\mathcal{P}_k} &= \vec{e}_{\mathcal{P}_k} \cdot \vec{\mathcal{P}}_k, \\ \mathcal{G}_{\mathcal{Q}_k} &= \vec{e}_{\mathcal{Q}_k} \cdot \vec{\mathcal{P}}_k, \\ \mathcal{G}_{\mathcal{R}_k} &= \vec{e}_{\mathcal{R}_k} \cdot \vec{\mathcal{P}}_k, \quad \text{for } k = 1, 2, \dots, n, \end{aligned} \quad (2.11)$$

where $\{\vec{e}_{\mathcal{P}_k}, \vec{e}_{\mathcal{Q}_k}, \vec{e}_{\mathcal{R}_k}\}$ are three-dimensional orthogonal unit vectors. One might think that the transformation (2.10) is merely a change of the basis of the $su(2)$ generators and does not change physics. This is not, however, the case. It should be emphasized that the transformation (2.10) is a *singular* unitary transformation because it, in general, changes connection conditions of wavefunctions at singular points. The transformation may be regarded as a duality connecting different theories (with different connection conditions).

3 $N = 2n$ superalgebra

In Ref. [10], $N = 2$ supercharges are constructed in the system on a circle with two point singularities placed at $x = 0$ and l . An extension of the supercharges to $N = 2n$ supercharges for any integer n will be given by

$$Q_a = \frac{i}{2} \Gamma_a \mathcal{D}, \quad \text{for } a = 1, 2, \dots, 2n, \quad (3.1)$$

where

$$\mathcal{D} = \mathcal{R}_1 \cdots \mathcal{R}_n \frac{d}{dx} + \mathcal{G}_{\mathcal{R}_1} \cdots \mathcal{G}_{\mathcal{R}_n} \mathcal{R}_1 \cdots \mathcal{R}_n W'(x), \quad (3.2)$$

$$\Gamma_{2k-1} = \mathcal{G}_{\mathcal{R}_1} \cdots \mathcal{G}_{\mathcal{R}_{k-1}} \mathcal{G}_{\mathcal{P}_k}, \quad (3.3)$$

$$\Gamma_{2k} = \mathcal{G}_{\mathcal{R}_1} \cdots \mathcal{G}_{\mathcal{R}_{k-1}} \mathcal{G}_{\mathcal{Q}_k}, \quad \text{for } k = 1, 2, \dots, n. \quad (3.4)$$

Here, $W'(x) = \frac{d}{dx} W(x)$ and $W(x)$ is called a superpotential. The function $W'(x)$ is allowed to have discontinuities at singular points $x = l_s$ ($s = 0, 1, \dots, 2^n - 1$) and is assumed to obey

$$\mathcal{P}_k W'(x) = -W'(x) \mathcal{P}_k, \quad \text{for } k = 1, 2, \dots, n. \quad (3.5)$$

Noting that $\mathcal{R}_1 \cdots \mathcal{R}_n \frac{d}{dx}$ and $\mathcal{R}_1 \cdots \mathcal{R}_n W'(x)$ commute with $\vec{\mathcal{P}}_k$ for $\forall k = 1, 2, \dots, n$, we can show that the supercharges Q_a ($a = 1, 2, \dots, 2n$) form the $N = 2n$ superalgebra

$$\{Q_a, Q_b\} = H \delta_{ab}, \quad \text{for } a, b = 1, 2, \dots, 2n \quad (3.6)$$

with the Hamiltonian

$$H = \frac{1}{2} \left[-\frac{d^2}{dx^2} - \mathcal{G}_{\mathcal{R}_1} \cdots \mathcal{G}_{\mathcal{R}_n} W''(x) + (W'(x))^2 \right]. \quad (3.7)$$

4 Compatibility with supersymmetry

It is important to realize that our quantum system is specified by not only the Hamiltonian but connection conditions for wavefunctions. This is because the system contains point singularities and we need to impose appropriate connection conditions there. The Hilbert space is then defined by a space spanned by eigenfunctions of the Hamiltonian (3.7) with the connection conditions which have to make the Hamiltonian hermitian. This setting is not, however, enough to guarantee the $N = 2n$ supersymmetry of the theory, because the hermiticity of the Hamiltonian does not, in general, assure the hermiticity of the supercharges Q_a ($a = 1, 2, \dots, 2n$) and further because for any state $\varphi(x)$ of the Hilbert space $Q_a \varphi(x)$ do not, in general, belong to the same Hilbert space (i.e., $Q_a \varphi(x)$ do not, in general, obey the same connection conditions as $\varphi(x)$). The supercharges would be then ill defined on the Hilbert space.

To give allowed connection conditions compatible with the $N = 2n$ supersymmetry, let us introduce a 2^{n+1} -dimensional vector Φ_φ that consists of boundary values of a wavefunction $\varphi(x)$ at the singularities, i.e., $\varphi(l_s \pm \varepsilon)$ for $s = 0, 1, \dots, 2^n - 1$ with an infinitesimal positive constant ε . It is convenient to arrange $\varphi(l_s \pm \varepsilon)$ in such a way that Φ_φ satisfies the relations

$$\Phi_{\mathcal{P}_k \varphi} = \overbrace{(I_2 \otimes \cdots \otimes I_2 \otimes \sigma_1 \otimes I_2 \otimes \cdots \otimes I_2)}^k \Phi_\varphi, \quad (4.1)$$

$$\Phi_{\mathcal{Q}_k \varphi} = (I_2 \otimes \cdots \otimes I_2 \otimes \sigma_2 \otimes I_2 \otimes \cdots \otimes I_2) \Phi_\varphi, \quad (4.2)$$

$$\Phi_{\mathcal{R}_k \varphi} = \underbrace{(I_2 \otimes \cdots \otimes I_2 \otimes \sigma_3 \otimes I_2 \otimes \cdots \otimes I_2)}_{n+1} \Phi_\varphi, \quad (4.3)$$

where I_M denotes an $M \times M$ unit matrix. For instance, Φ_φ for $n = 1$ will be given by

$$\Phi_\varphi = (\varphi(l - \varepsilon), \varphi(0 + \varepsilon), \varphi(-l + \varepsilon), \varphi(0 - \varepsilon))^T. \quad (4.4)$$

For $n = 2$, Φ_φ will be given by

$$\begin{aligned} \Phi_\varphi = & (\varphi(l - \varepsilon), \varphi(l/2 + \varepsilon), \varphi(0 + \varepsilon), \varphi(l/2 - \varepsilon), \\ & \varphi(-l + \varepsilon), \varphi(-l/2 - \varepsilon), \varphi(0 - \varepsilon), \varphi(-l/2 + \varepsilon))^T. \end{aligned} \quad (4.5)$$

Let us consider the following type of connection conditions for a wavefunction $\varphi(x)$:

$$(I_{2^{n+1}} - U) \Phi_\varphi = 0, \quad (4.6)$$

$$(I_{2^{n+1}} + U) \Sigma_3 \Phi_{\mathcal{D}\varphi} = 0, \quad (4.7)$$

where U is any $2^{n+1} \times 2^{n+1}$ matrix satisfying

$$\Sigma_3 \gamma_a U = -U \Sigma_3 \gamma_a, \quad \text{for } a = 1, 2, \dots, 2n, \quad (4.8)$$

$$U^\dagger U = I_{2^{n+1}}, \quad (4.9)$$

$$U^2 = I_{2^{n+1}}, \quad (4.10)$$

with

$$\Sigma_3 = I_2 \otimes \dots \otimes I_2 \otimes \sigma_3, \quad (4.11)$$

$$\gamma_{2k-1} = \vec{e}_{\mathcal{R}_1} \cdot \vec{\sigma} \otimes \dots \otimes \vec{e}_{\mathcal{R}_{k-1}} \cdot \vec{\sigma} \otimes \vec{e}_{\mathcal{P}_k} \cdot \vec{\sigma} \otimes I_2 \otimes \dots \otimes I_2, \quad (4.12)$$

$$\gamma_{2k} = \vec{e}_{\mathcal{R}_1} \cdot \vec{\sigma} \otimes \dots \otimes \vec{e}_{\mathcal{R}_{k-1}} \cdot \vec{\sigma} \otimes \vec{e}_{\mathcal{Q}_k} \cdot \vec{\sigma} \otimes I_2 \otimes \dots \otimes I_2, \quad (4.13)$$

for $k = 1, 2, \dots, n$.

The Hilbert space is then assumed to be spanned by eigenfunctions of the Hamiltonian (3.7) satisfying the connection conditions (4.6) and (4.7). The last condition (4.10) implies that any eigenvalue of U is $+1$ or -1 , so that the total number of the constraints in Eqs. (4.6) and (4.7) is 2×2^n . This is the correct number to solve the Schrödinger equation in our system because two connection conditions between $\varphi(x)$ and $\varphi'(x)$ should be imposed at each point singularity and there are 2^n point singularities in the present model.

The hermiticity conditions of the supercharges

$$\int_{-l}^l dx \psi^*(x) (Q_a \varphi)(x) = \int_{-l}^l dx (Q_a \psi)^*(x) \varphi(x), \quad \text{for } a = 1, 2, \dots, 2n \quad (4.14)$$

give the nontrivial constraints on boundary values of the wavefunctions

$$\Phi_\psi^\dagger \Sigma_3 \gamma_a \Phi_\varphi = 0, \quad \text{for } a = 1, 2, \dots, 2n. \quad (4.15)$$

To derive it, we may use the formula of integration by parts

$$\int_{-l}^l dx \xi^*(x) \left(\frac{d}{dx} \eta(x) \right) = - \int_{-l}^l dx \left(\frac{d}{dx} \xi(x) \right)^* \eta(x) + \Phi_\xi^\dagger (\sigma_3 \otimes \dots \otimes \sigma_3) \Phi_\eta, \quad (4.16)$$

where the functions $\xi(x)$ and $\eta(x)$ are assumed to be continuous everywhere except for point singularities. It is easy to show that the conditions (4.15) are satisfied if $\varphi(x)$ and $\psi(x)$ obey the connection conditions (4.6) and (4.7). We can further show that for any eigenfunction $\varphi(x)$ of the Hilbert space $Q_a \varphi(x)$ ($a = 1, 2, \dots, n$) also obey the connection conditions (4.6) and (4.7). This implies that for any state $\varphi(x)$ of the Hilbert space any products of Q_a 's on $\varphi(x)$, $Q_{a_1} Q_{a_2} \dots Q_{a_m} \varphi(x)$, belong to the same Hilbert space as $\varphi(x)$. It follows from the algebra (3.6) that the Hamiltonian is hermitian, as it should be. Therefore, the action of the supercharges on the Hilbert space is well-defined, and the algebra (3.6) guarantees the $N = 2n$ supersymmetry of the theory.

It turns out that the following two types of the matrix U satisfy the desired relations:

(I) Type I

$$U_I(\pm) = \pm (\vec{e}_{\mathcal{R}_1} \cdot \vec{\sigma} \otimes \cdots \otimes \vec{e}_{\mathcal{R}_n} \cdot \vec{\sigma} \otimes I_2), \quad (4.17)$$

(II) Type II

$$\begin{aligned} U_{II}(a) = & a_1 (I_2 \otimes \cdots \otimes I_2 \otimes \sigma_1) + a_2 (I_2 \otimes \cdots \otimes I_2 \otimes \sigma_2) \\ & + a_3 (\vec{e}_{\mathcal{R}_1} \cdot \vec{\sigma} \otimes \cdots \otimes \vec{e}_{\mathcal{R}_n} \cdot \vec{\sigma} \otimes \sigma_3) \end{aligned} \quad (4.18)$$

with $a_1, a_2, a_3 \in \mathbf{R}$ and $(a_1)^2 + (a_2)^2 + (a_3)^2 = 1$. The connection conditions found in Ref. [10] correspond to the type I and the type II solutions with $a_1 = a_2 = 0$ for $n = 1$. Although the configuration spaces in Ref. [9, 11] are different from ours, the results seem to be consistent with ours for $n = 1$ and $n = 2$ with a free Hamiltonian. In our derivation, it is unclear whether the solutions (4.17) and (4.18) exhaust all allowed connection conditions compatible with supersymmetry. This issue will be discussed in a forthcoming paper [13].

5 Degeneracy of the spectrum

In the previous sections, we have succeeded to construct the $N = 2n$ supercharges and found the connection conditions compatible with supersymmetry. In this section, we study the degeneracy of the spectrum, in particular, vacuum states with vanishing energy.

We first note that $\mathcal{G}_{\mathcal{R}_k}$ ($k = 1, 2, \dots, n$) commute with H and also with each other. These facts guarantee that we can introduce simultaneous eigenfunctions of the Hamiltonian and $\mathcal{G}_{\mathcal{R}_k}$ ($k = 1, 2, \dots, n$) such that

$$H \varphi_{E; \lambda_1, \dots, \lambda_n}(x) = E \varphi_{E; \lambda_1, \dots, \lambda_n}(x), \quad (5.1)$$

$$\mathcal{G}_{\mathcal{R}_k} \varphi_{E; \lambda_1, \dots, \lambda_n}(x) = \lambda_k \varphi_{E; \lambda_1, \dots, \lambda_n}(x) \quad (5.2)$$

with $\lambda_k = 1$ or -1 for $k = 1, 2, \dots, n$. Since Q_a ($a = 1, 2, \dots, 2n$) and $\mathcal{G}_{\mathcal{R}_k}$ ($k = 1, 2, \dots, n$) satisfy the relations

$$Q_a \mathcal{G}_{\mathcal{R}_k} = \begin{cases} -\mathcal{G}_{\mathcal{R}_k} Q_a & \text{if } a = 2k - 1 \text{ or } 2k, \\ +\mathcal{G}_{\mathcal{R}_k} Q_a & \text{if otherwise,} \end{cases} \quad (5.3)$$

the states $Q_{2k-1} \varphi_{E; \lambda_1, \dots, \lambda_n}(x)$ and $Q_{2k} \varphi_{E; \lambda_1, \dots, \lambda_n}(x)$ should be proportional to $\varphi_{E; \lambda_1, \dots, -\lambda_k, \dots, \lambda_n}(x)$, i.e.,

$$Q_{2k-1} \varphi_{E; \lambda_1, \dots, \lambda_n}(x) = -i \lambda_k Q_{2k} \varphi_{E; \lambda_1, \dots, \lambda_n}(x) \propto \varphi_{E; \lambda_1, \dots, -\lambda_k, \dots, \lambda_n}(x), \quad (5.4)$$

when $E \neq 0$. This implies that the degeneracy of the spectrum for $E \neq 0$ is given by 2^n . This result can be obtained from an algebraic point of view; for a fixed nonzero energy E , Q_a/\sqrt{E} for $a = 1, 2, \dots, 2n$ form the Clifford algebra, and the representation is known as 2^n .

The above argument cannot apply for states with $E = 0$. This is because any state $\varphi_0(x)$ with vanishing energy satisfies

$$Q_a \varphi_0(x) = 0 \quad \text{for } \forall a = 1, 2, \dots, 2n. \quad (5.5)$$

It is easy to show that there are 2^n formal solutions to the above equations

$$\varphi_{0;\lambda_1,\dots,\lambda_n}(x) = N_{\lambda_1\dots\lambda_n} \left[\prod_{k=1}^n \frac{1}{2} (1 + \lambda_k \mathcal{G}_{\mathcal{R}_k}) \right] e^{-\lambda_1 \dots \lambda_n W(x)} \quad (5.6)$$

with $\lambda_k = 1$ or -1 for $k = 1, 2, \dots, n$. Here, $N_{\lambda_1\dots\lambda_n}$ denote normalization constants. For a noncompact space, any non-normalizable states would be removed from the Hilbert space. The space is, however, compact (a circle) in our model, so that the solutions (5.6) are always normalizable. Nevertheless, some of them must be removed from the Hilbert space. This occurs due to incompatibility with the connection conditions (4.17) and (4.18).

The zero energy states for the type I connection conditions with $U_1(+)$ ($U_1(-)$) are given by $\varphi_{0;\lambda_1,\dots,\lambda_n}(x)$ with $\lambda_1 \lambda_2 \dots \lambda_n = +1$ (-1). The remaining states with $\lambda_1 \lambda_2 \dots \lambda_n = -1$ ($+1$) do not satisfy the connection conditions, and hence they must be removed from the Hilbert space. Thus, the zero energy vacua are 2^{n-1} -fold degenerate, and supersymmetry is unbroken⁴. For the type II connection conditions, all the states (5.6) are found to be inconsistent with the connection conditions, so that there are no vacuum states with zero energy. There is, however, an exception. If the following relations are satisfied

$$\begin{aligned} \sqrt{\frac{1-a_3}{1+a_3}} &= e^{W(l_0)-W(l_1)}, \\ a_1 &= \sqrt{1-(a_3)^2}, \\ a_2 &= 0, \end{aligned} \quad (5.7)$$

all the states (5.6) accidentally become supersymmetric vacuum states compatible with the connection conditions. Therefore, for the type II connection conditions, supersymmetry is spontaneously broken except for the above case.

6 Summary and discussions

In this Letter, we have constructed the $N = 2n$ supercharges and found a class of the connection conditions compatible with supersymmetry in one-dimensional quantum mechanics on a circle with 2^n point singularities. The supercharges are represented in terms of the discrete transformations $\{\mathcal{P}_k, \mathcal{Q}_k, \mathcal{R}_k\}$ ($k = 1, 2, \dots, n$). The action of $\{\mathcal{P}_k, \mathcal{Q}_k, \mathcal{R}_k\}$, in general, makes wavefunctions discontinuous, so that our realization of the $N = 2n$ supersymmetry reflects the characteristics of singularities in quantum mechanics.

⁴By analogy with supersymmetric quantum field theory, we say that supersymmetry is spontaneously broken if the action of the supercharges on any vacuum is nonvanishing.

In our analysis, we required that all the $2n$ supercharges are hermitian and well-defined on the Hilbert space. We can, instead, require that only a subset of them are hermitian and well-defined to reduce the $N = 2n$ supersymmetry. In other words, we allow some of the $2n$ supercharges to become ill defined due to connection conditions. This implies that the introduction of a number of point singularities can lead to a wide variety of N -extended supersymmetric models for any integer N .

It is interesting to notice that there exists one more discrete transformation \mathcal{P}_{n+1} that produces singularities at $x = l_s$ for $s = 0, 1, \dots, 2^n - 1$ but no other points. Adding \mathcal{P}_{n+1} to the algebra, we can construct $2n + 1$ supercharges that form an $N = 2n + 1$ superalgebra [13]. Any subset of the $2n + 1$ supercharges does not, however, coincide with the $N = 2n$ supercharges in Eqs. (3.1) (unless the Hamiltonian is free). Thus, the $N = 2n + 1$ supersymmetry including \mathcal{P}_{n+1} in the algebra belongs to a different class from the $N = 2n$ supersymmetry considered in this Letter. Full details will be discussed in a forthcoming paper [13].

Acknowledgements

The authors wish to thank C.S. Lim, S. Tanimura, I. Tsutsui for valuable discussions and useful comments. This work was supported in part by the Grant-in-Aid for Scientific Research (No.15540277) by the Japanese Ministry of Education, Science, Sports and Culture. K.T. is supported by The 21st century COE program named “Towards a new basic science : depth and synthesis”.

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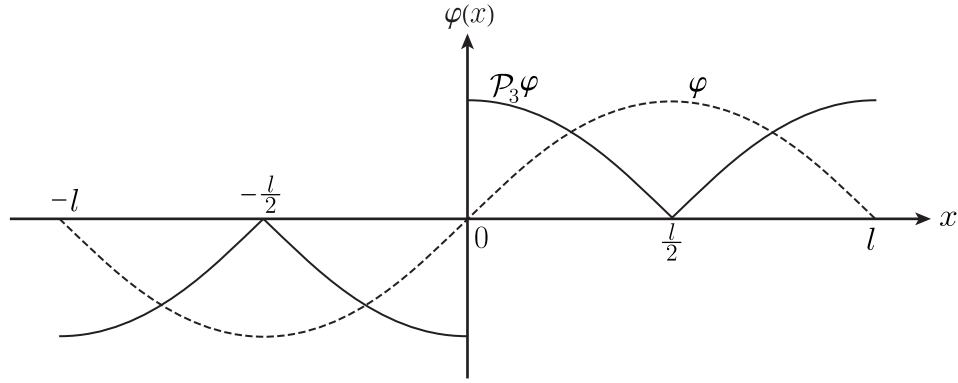


Figure 1: An example of the action of \mathcal{P}_3 on a function $\varphi(x)$. The dashed line denotes the original function $\varphi(x)$, and the solid line denotes $(\mathcal{P}_3\varphi)(x)$.

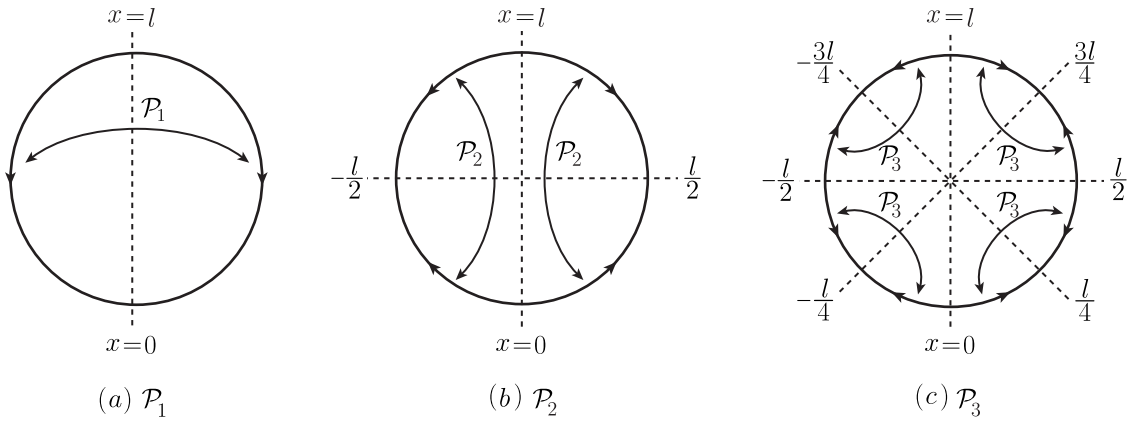


Figure 2: The geometrical meanings of \mathcal{P}_k for $k = 1, 2$ and 3 .

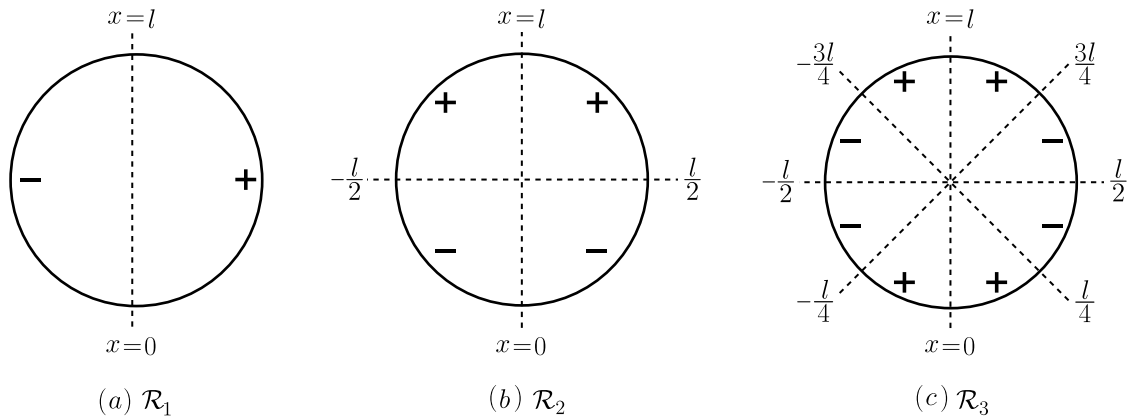


Figure 3: The geometrical meanings of \mathcal{R}_k for $k = 1, 2$, and 3 .